2.3 Basic Differentiation Rules

$\frac{d}{dx}[f(x)] \rightarrow$ the derivative of $f(x)$

1) $\frac{d}{dx}[c] = 0$

1(b) $\frac{d}{dx}[cx] = c$

2) $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$

3) $\frac{d}{dx}[x^n] = nx^{n-1}$

4) $\frac{d}{dx}[ax^n] = nax^{n-1}$

Power Rule

5) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

6) $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

$$g(x) = \begin{cases} mx^2 - 7x + 2n, & \text{if } x < 2 \\ \frac{x+2}{x}, & \text{if } x = 2 \\ \sqrt{x+n}, & \text{if } x > 2 \end{cases}$$

for continuity at $x = 2$, $\Rightarrow$ $f(2) = \lim_{x \to 2} g(x)$

or $\Rightarrow$ $f(2) = \lim_{x \to 2^+} g(x)$ and $f(2) = \lim_{x \to 2^-} g(x)$

$$\frac{2+2}{2} = \sqrt{2+n}$$

$$2 = \sqrt{2+n}$$

$$4 = 2+n$$

$$2 = n$$

$$\frac{2+2}{2} = 4m \cdot 14 + 2n$$

$$2 = 4m - 14 + 2(2)$$

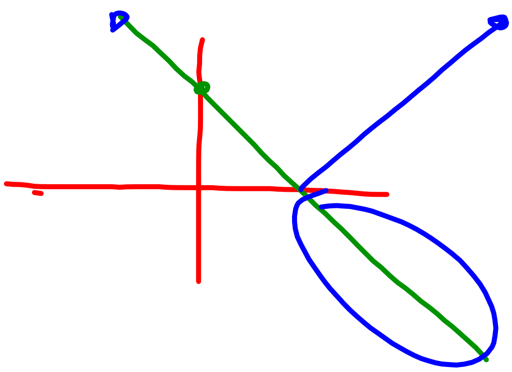
$$2 = 4m - 10$$

$$12 = 4m \quad m = 3$$

$$\lim_{x \to 3^-} \frac{|3-x|}{x^2-4x+3}$$

$$|3-x| = \begin{cases} 3-x, & x \le 3 \\ -(3-x), & x > 3 \end{cases}$$

$$\lim_{x \to 3^-} \frac{3-x}{(x-3)(x-1)}$$



$$\lim_{x \to 3^-} \frac{-1}{x-1} = -\frac{1}{2}$$

$$\lim_{x \to 0} \frac{(3+x)^{-1} - 3^{-1}}{x}$$

$$\lim_{x \to 0} \frac{\left(\frac{1}{3+x} - \frac{1}{3}\right)}{(x)} \frac{(3(3+x))}{(3(3+x))}$$

$$\lim_{x \to 0} \frac{3 - (3+x)}{x(3(3+x))}$$

$$\lim_{x \to 0} \frac{\cancel{3} - \cancel{3} - x}{x(3(3+x))}$$

$$-\frac{1}{3(3+0)}$$

$$= -\frac{1}{9}$$

2.4 The Product and Quotient Rules

Product Rule:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + f'(x)\cdot g(x)$$

Ex: $$\frac{d}{dx}\left[\sqrt{x}\cdot(3x^2+1)\right] =$$

$\sqrt{x} = x^{\frac{1}{2}}$

$\frac{d}{dx}[x^{\frac{1}{2}}]$

$\frac{1}{2}x^{\frac{1}{2}-1}$

$$= \overbrace{\sqrt{x}}^{f(x)}\,\overbrace{\frac{d}{dx}(3x^2+1)}^{g'(x)} + \overbrace{\frac{d}{dx}[\sqrt{x}]}^{f'(x)}\,\overbrace{(3x^2+1)}^{g}$$

$$= \sqrt{x}(6x) + \frac{1}{2}x^{-\frac{1}{2}}(3x^2+1)$$

$$= \sqrt{x}(6x) + \frac{1}{2\sqrt{x}}\,\frac{(3x^2+1)}{1}$$

$$= 6x\sqrt{x} + \frac{3x^2+1}{2\sqrt{x}}$$

$$= \frac{(6x\sqrt{x})(2\sqrt{x})}{2\sqrt{x}} + \frac{3x^2+1}{2\sqrt{x}}$$

$$= \frac{12x^2 + 3x^2+1}{2\sqrt{x}} = \frac{15x^2+1}{2\sqrt{x}}$$

Ex: $\frac{d}{dx}\left[\overset{f}{\sqrt{x}} \cdot \overset{g}{(3x^2+1)}\right] = \overset{f}{\sqrt{x}} \cdot \overset{g'}{\frac{d}{dx}[3x^2+1]} + \overset{g}{(3x^2+1)} \overset{f'}{\frac{d}{dx}[\sqrt{x}]}$

$= \sqrt{x}\,(6x) + (3x^2+1)\boxed{\frac{d}{dx}\left(x^{\frac{1}{2}}\right)}$

$= 6x\sqrt{x} + (3x^2+1)\left[\frac{1}{2}x^{-\frac{1}{2}}\right]$

$= \frac{2\sqrt{x}}{2\sqrt{x}}\boxed{6x\sqrt{x} + (3x^2+1)\left(\frac{1}{2\sqrt{x}}\right)}$

$= \frac{12x^2 + 3x^2 + 1}{2\sqrt{x}}$

$= \frac{15x^2+1}{2\sqrt{x}}$

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + f'(x)\,g(x)$$

$$[f(x)g(x)]' = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

The trick now is to introduce a zero term; $-f(x+h)g(x)+f(x+h)g(x)$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\left[g(x+h)-g(x)\right]}{h} + \lim_{h\to 0}\frac{g(x)\left[f(x+h)-f(x)\right]}{h}$$

$$= \lim_{h\to 0} f(x+\cancel{h}^{\,0})\cdot \underbrace{\boxed{\lim_{h\to 0}\frac{\left[g(x+h)-g(x)\right]}{h}}}_{g'(x)} + \lim_{h\to 0} g(x)\,\overbrace{\lim_{h\to 0}\frac{\left[f(x+h)-f(x)\right]}{h}}^{f'(x)}$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Ex: $\frac{d}{dx}\left[\sqrt{x}\cdot(3x^2+1)\right]$

$$\frac{d}{dx}\left[\sqrt{x}\cdot(3x^2+1)\right] = \overset{f}{\sqrt{x}}\cdot\overbrace{\frac{d}{dx}\left[3x^2+1\right]}^{g'} + \overbrace{(3x^2+1)}^{g}\cdot\overbrace{\frac{d}{dx}\left[\sqrt{x}\right]}^{f'}$$

$$= \sqrt{x}(6x) + (3x^2+1)\frac{d}{dx}\left[x^{\frac{1}{2}}\right]$$

$$= 6x\sqrt{x} + (3x^2+1)\left(\frac{1}{2\sqrt{x}}\right)$$

$$= 6x\sqrt{x} + \frac{3x^2+1}{2\sqrt{x}}$$

$$= 6x\overbrace{\sqrt{x}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}}^{=x} + \frac{3x^2+1}{2\sqrt{x}}$$

$$= \frac{12x^2+3x^2+1}{2\sqrt{x}}$$

$$= \frac{15x^2+1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + f'(x)\,g(x)$$

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0}\frac{f(x+h)\,g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h}\right]$$

$$= \lim_{h\to 0} f(x+h)\cdot\left[\frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0} g(x)\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= \underbrace{\lim_{h\to 0} f(x+\cancel{h})}_{f(x)}\cdot\boxed{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}} + \underbrace{\lim_{h\to 0} g(x)}_{g(x)}\cdot\boxed{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

$$f(x)\cdot g'(x) \quad + \quad g(x)\cdot f'(x)$$

Quotient Rule:

for $\frac{f(x)}{g(x)}$, $g(x) \neq 0$ and f and g are differentiable

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left[g(x)\right]^2}$$

Ex: $$\frac{d}{dx}\left[\frac{x^2+5x}{\sqrt{x}}\right] = \frac{\sqrt{x}\,\overset{\frac{d}{dx}(x^2+5x)}{(2x+5)} - (x^2+5x)\overset{\frac{d}{dx}(\sqrt{x})}{\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}}{(\sqrt{x})^2}$$

$$= \frac{\left[\sqrt{x}(2x+5) - (x^2+5x)\left(\frac{1}{2\sqrt{x}}\right)\right]}{x} \cdot \frac{(2\sqrt{x})}{(2\sqrt{x})}$$

$$= \frac{2x(2x+5) - (x^2+5x)(1)}{2x\sqrt{x}}$$

$$= \frac{4x^2+10x - x^2 - 5x}{2x\sqrt{x}}$$

$$= \frac{3x^2+5x}{2x\sqrt{x}}$$

$$= \frac{\cancel{x}(3x+5)}{2\cancel{x}\sqrt{x}} = \frac{3x+5}{2\sqrt{x}}$$

1. Calculate the derivative of $f(x)=(1+2x^2)(x-x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

Product Rule:

$$f'(x) = (1+2x^2)\frac{d}{dx}(x-x^2) + (x-x^2)\frac{d}{dx}[1+2x^2]$$

$$= (1+2x^2)(1-2x) + (x-x^2)(0+4x)$$

$$= 1-2x+2x^2-4x^3+4x^2-4x^3$$

$$f'(x) = -8x^3+6x^2-2x+1$$

Simplify first: $f(x)=(1+2x^2)(x-x^2)$

$$f(x) = x-x^2+2x^3-2x^4$$

$$f(x) = -2x^4+2x^3-x^2+x$$

$$f'(x) = -8x^3+6x^2-2x+1$$

Same ☺

(F) $g(x)=x^2\left(\frac{2}{x}-\frac{1}{x+1}\right)$

$$g(x)=2x-\frac{x^2}{x+1}$$

$$g'(x)=2-\frac{(x+1)\frac{d}{dx}(x^2)-(x^2)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$g'(x)=2-\frac{(x+1)(2x)-(x^2)(1+0)}{(x+1)^2}$$

$$g'(x)=2-\frac{2x^2+2x-x^2}{(x+1)^2}$$

$$g'(x)=2\frac{(x+1)^2}{(x+1)^2}-\frac{x^2+2x}{(x+1)^2}$$

$$g'(x)=\frac{2(x^2+2x+1)-(x^2+2x)}{(x+1)^2}$$

$$g'(x)=\frac{2x^2+4x+2-x^2-2x}{(x+1)^2}$$

$$g'(x)=\frac{x^2+2x+2}{(x+1)^2}$$

Ex: $\frac{d}{dx}\left[\frac{x^2+5x}{\sqrt{x}}\right]$

$$\frac{d}{dx}\left[\frac{x^2+5x}{\sqrt{x}}\right] = \frac{\sqrt{x}\cdot\frac{d}{dx}(x^2+5x) - (x^2+5x)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}\,(2x+5) - (x^2+5x)\left(\frac{1}{2\sqrt{x}}\right)}{x}$$

$$= \frac{\left(\sqrt{x}\,(2x+5) - \frac{x^2+5x}{2\sqrt{x}}\right)}{x}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{2\sqrt{x}\cdot\sqrt{x}\,(2x+5) - (x^2+5x)}{2x^{\frac{1}{2}}\cdot x^1}$$

$$= \frac{4x^2+10x - (x^2+5x)}{2x^{\frac{3}{2}}}$$

$$= \frac{3x^2+5x}{2x^{\frac{3}{2}}} = \frac{3x+5}{2\sqrt{x}}$$

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#'s 1, 2, 5, 6, 11-18, 27, 28

P. 120 # 1, 2, 7, 8, 9, 11, 17-24

The Chain Rule (section 2.5 in text)

→ used when we have compositions of functions

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

Ex: $\frac{d}{dx}\left[(3x^2+1)^{50}\right] = 50(3x^2+1)^{49} \cdot \frac{d}{dx}(3x^2+1)$

$= 50(3x^2+1)^{\overset{50-1}{49}} \cdot 6x$

$= 300x(3x^2+1)^{49}$

Ex ② $\frac{d}{dx}\left[\sqrt{5x^2+2x}\right]$

$\frac{d}{dx}\left[(5x^2+2x)^{\frac{1}{2}}\right] = \frac{1}{2}(5x^2+2x)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(5x^2+2x)$

$= \frac{1}{2}(5x^2+2x)^{-\frac{1}{2}}(10x+2)$

$= \dfrac{10x+2}{2\sqrt{5x^2+2x}}$

$= \dfrac{5x+1}{\sqrt{5x^2+2x}}$

$$\frac{d}{dx}\left[f(g(h(k(x))))\right]$$

$$= f'(g(h(k(x)))) \cdot g'(h(k(x))) \cdot h'(k(x)) \cdot k'(x)$$

Ex: $\frac{d}{dx}\left[\sqrt{(2x^3+x)^{25}}\right]$

$$\frac{d}{dx}\left(\left(2x^3+x\right)^{25}\right)^{\frac{1}{2}}$$

$$\frac{1}{2}\left(\underbrace{\left(2x^3+x\right)^{25}}_{g}\right)^{-\frac{1}{2}} \underbrace{25\left(\boxed{2x^3+x}_{h}\right)^{24}}_{g'} \cdot \underbrace{\left(6x^2+1\right)}_{h'}$$

$$\frac{25\left(6x^2+1\right)\left(2x^3+x\right)^{24}}{2\left(2x^3+x\right)^{\frac{25}{2}}}$$

$$\frac{25\left(6x^2+1\right)\left(2x^3+x\right)^{24-\frac{25}{2}}}{2}$$

$$\frac{25\left(6x^2+1\right)\left(2x^3+x\right)^{\frac{23}{2}}}{2}$$

$$\frac{d}{dx}\left(2x^3+x\right)^{\frac{25}{2}}$$

$$\frac{25}{2}\left(2x^3+x\right)^{\frac{25}{2}-1} \cdot \frac{d}{dx}\left(2x^3+x\right)$$

$$\frac{25}{2}\left(2x^3+x\right)^{\frac{23}{2}}\left(6x^2+1\right)$$

$$\frac{25\left(6x^2+1\right)\left(2x^3+x\right)^{\frac{23}{2}}}{2}$$

Ex: $\frac{d}{dx}\left[\left(\frac{x^2-1}{x^2+1}\right)^3\right]$

$$= 3\left(\frac{x^2-1}{x^2+1}\right)^2 \cdot \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$\frac{d}{dx}[x^2-1]$ $\frac{d}{dx}[x^2+1]$

$$= 3\left(\frac{x^2-1}{x^2+1}\right)^2 \cdot \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= 3\,\frac{(x^2-1)^2}{(x^2+1)^2}\,\frac{\left(\cancel{2x^3} + 2x - \cancel{2x^3} + 2x\right)}{(x^2+1)^2}$$

$$= \frac{12x(x^2-1)^2}{(x^2+1)^4}$$

P.120 # 1, 2, 7, 8, 9, 11, 17-24

$$\frac{d}{dx}\left[x^8\right] = 8x^7$$

$$\frac{d}{dx}\left[3x^2+2x\right] = 6x+2$$

$$\frac{d}{dx}\left[(3x^2+2x)^8\right] = 8(3x^2+2x)^7(6x+2)$$

$$= 8(6x+2)(3x^2+2x)^7$$

$$= 16(3x+1)(3x^2+2x)^7$$

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\cdot g'(x)$$

$$y = (6x^2+3)^8(2x^4-5x^2+1)^4$$

$$y' = \frac{d}{dx}\left[(6x^2+3)^8\right]\left[(2x^4-5x^2+1)^4\right] + \frac{d}{dx}\left[(2x^4-5x^2+1)^4\right]\left[(6x^2+3)^8\right]$$

$$y' = \underbrace{8(6x^2+3)^7(12x)}_{\text{chain rule}}(2x^4-5x^2+1)^4 + \underbrace{4(2x^4-5x^2+1)^3(8x^3-10x)}_{\text{chain rule}}(6x^2+3)^8$$

$$y' = 4(6x^2+3)^7(2x^4-5x^2+1)^3\left[24x(2x^4-5x^2+1) + (8x^3-10x)(6x^2+3)\right]$$

$$f(x) = \frac{2x+5}{\sqrt{2-x^2}} \Rightarrow f(x) = \frac{2x+5}{(2-x^2)^{\frac{1}{2}}}$$

$$f'(x) = \frac{(2-x^2)^{\frac{1}{2}}(2) - (2x+5)\overbrace{\frac{1}{\cancel{2}}(2-x^2)^{-\frac{1}{2}}(-\cancel{2}x)}^{\text{chain rule}}}{\left[(2-x^2)^{\frac{1}{2}}\right]^{\cancel{2}}}$$

$$f'(x) = \frac{\left(2\sqrt{2-x^2} + \frac{x(2x+5)}{\cancel{\sqrt{2-x^2}}}\right)}{(2-x^2)^1} \cdot \frac{\cancel{\sqrt{2-x^2}}}{\sqrt{2-x^2}}$$

$$f'(x) = \frac{2(2-x^2) + x(2x+5)}{(2-x^2)^{\frac{3}{2}}}$$

$$f'(x) = \frac{4 - \cancel{2x^2} + \cancel{2x^2} + 5x}{(2-x^2)^{3/2}}$$

$$f'(x) = \frac{5x+4}{(2-x^2)^{3/2}}$$

$$\frac{d}{dx}\left[(x^2+1)^{10}\right] = 10(x^2+1)^9 \cdot \frac{d}{dx}\left[x^2+1\right]$$

$$\frac{d}{dx}\left[(y)^{10}\right] = 10(y)^9 \cdot \frac{d}{dx}\left[y\right]$$

$$= 10(y)^9 \cdot y'$$

$$\frac{d}{dx}\left[x^2+y^2\right] = 2x + 2y \cdot y'$$