Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\left(\sin^{-1}x\right)$$

let $y = \sin^{-1}x$

then $\sin y = x$

OR $x = \sin y$

Now evaluate $\frac{d}{dx}\left[x = \sin y\right]$

$$\frac{1}{\cos y} = \frac{\cos y \cdot y'}{\cos y}$$

$$y' = \frac{1}{\cos y}$$

Recall $\cos^2 y + \sin^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1-\sin^2 y}$

in 1^{st} Quad and 4^{th} Quad

cos is positive

$\therefore \cos y = \sqrt{1-\sin^2 y}$

since $x = \sin y$

$\cos y = \sqrt{1-x^2}$

$$\boxed{\text{Thus, for } y = \sin^{-1}(x) \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

1. $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$

2. $\frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1-x^2}}$

3. $\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$

4. $\frac{d}{dx}\left[\cot^{-1} x\right] = -\frac{1}{1+x^2}$

5. $\frac{d}{dx}\left[\csc^{-1} x\right] = -\frac{1}{x\sqrt{x^2-1}}$

6. $\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{x\sqrt{x^2-1}}$

Ex:

#16 $y = \tan^{-1}(x^2)$ what is y'

$$\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$$

$$\text{so } \frac{d}{dx}\left[\tan^{-1}(x^2)\right] = \frac{1}{1+(x^2)^2} \cdot 2x = \boxed{\frac{2x}{1+x^4}}$$

$$\boxed{\text{i.e. } \frac{d}{dx}\left[\tan^{-1}(u)\right] = \frac{1}{1+u^2} \cdot \frac{du}{dx}}$$

#22. $F(\theta) = \arcsin\sqrt{\sin\theta}$

$$\text{Recall } \frac{d}{dx}\left[\arcsin x\right] = \frac{1}{\sqrt{1-x^2}}$$

$$\text{so } \frac{d}{dx}\left[\arcsin u\right] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\text{now } \frac{d}{d\theta}\left[\arcsin\sqrt{\sin\theta}\right] = \frac{1}{\sqrt{1-(\sqrt{\sin\theta})^2}} \cdot \frac{d}{d\theta}\left(\sqrt{\sin\theta}\right)$$

$(\sin)^{\frac{1}{2}}$

$$= \frac{1}{\sqrt{1-\sin\theta}} \cdot \frac{1}{2}(\sin\theta)^{-\frac{1}{2}} \cdot \cos\theta$$

$$= \frac{\cos\theta}{2\sqrt{\sin\theta}\sqrt{1-\sin\theta}} = \frac{\cos\theta}{2\sqrt{\sin\theta - \sin^2\theta}}$$

#31 $g(x) = x\sin^{-1}\left(\frac{x}{4}\right) + \sqrt{16-x^2}$

What is $g'(2)$?

$$g'(x) = x\left[\frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}\cdot\frac{1}{4}\right] + 1\cdot\sin^{-1}\left(\frac{x}{4}\right) + \frac{1}{\cancel{2}}(16-x^2)^{-\frac{1}{2}}(-\cancel{2}x)$$

$$g'(x) = \frac{x}{4\sqrt{1-\frac{x^2}{16}}} + \sin^{-1}\left(\frac{x}{4}\right) - \frac{x}{\sqrt{16-x^2}}$$

$$\text{So, } g'(2) = \frac{\cancel{2}}{\underset{2}{\cancel{4}}\sqrt{1-\frac{2^2}{16}}} + \sin^{-1}\left(\frac{2}{4}\right) - \frac{2}{\sqrt{16-2^2}}$$

$$= \frac{1}{2\sqrt{\frac{3}{4}}} + \frac{\pi}{6} - \frac{2}{\sqrt{12}} = \frac{\cancel{2}}{\cancel{2}\sqrt{3}}$$

$\cancel{2\frac{\sqrt{3}}{2}}$

$$= \cancel{\frac{1}{\sqrt{3}}} + \frac{\pi}{6} - \cancel{\frac{1}{\sqrt{3}}} = \frac{\pi}{6}$$

Inverse Trig Derivatives

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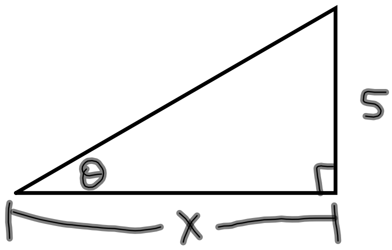
omit # 26

Related Rates → The Revenge!

Ex: An airplane flies at an altitude of 5 miles toward an observer. The speed of the plane is 600mph. At what rate is the angle of elevation changing when the angle measures 30°



$\frac{d\theta}{dt} = ?$

$\frac{dx}{dt} = -600$

$\tan\theta = \frac{5}{x}$

$\tan 30 = \frac{5}{x}$

$\frac{\sqrt{3}}{3} = \frac{5}{x}$

$\sqrt{3}x = 15$

$x = \frac{15}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$\boxed{x = 5\sqrt{3}}$

$$\tan\theta = \frac{5}{x}$$

$$\tan\theta = 5x^{-1}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = -5x^{-2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{5}{x^2\sec^2\theta}\frac{dx}{dt}$$

$\theta = 30°$

$$\frac{d\theta}{dt} = -\frac{5}{(5\sqrt{3})^2\left(\frac{2}{\sqrt{3}}\right)^2}(-600)$$

$$\frac{d\theta}{dt} = -\frac{5}{(75)\left(\frac{4}{3}\right)}(-600)$$

$$\frac{d\theta}{dt} = \frac{3000}{100} = 30°/\text{hr}$$

Optimization Problems

Ex: The position of a particle as it moves horizontally is described by the equation $s = 2\sin t + \sin 2t$, $-\pi \leq t \leq \pi$. If s is the displacement in metres and t is the time in seconds, determine the absolute maximum and absolute minimum displacements.

$$\frac{ds}{dt} = 2\cos t + (\cos 2t)2$$

$$\frac{ds}{dt} = 2\cos t + 2\cos 2t$$

Critical #s $\Rightarrow \frac{ds}{dt} = 0$

$$2\cos t + 2\cos 2t = 0$$

$$\cos t + \cos 2t = 0$$

$(2x^2 + x - 1)$

$$\cos t + \boxed{2\cos^2 t - 1} = 0$$

$$2\cos^2 t + \cos t - 1 = 0$$

$$(\cos t + 1)(2\cos t - 1) = 0$$

$\cos t + 1 = 0 \qquad 2\cos t - 1 = 0$

$\cos t = -1 \qquad \cos t = \frac{1}{2}$

$t = \pm\pi \qquad t = \frac{\pi}{3}, -\frac{\pi}{3}$

$$s = 2\sin t + \sin 2t$$

test $t = \pi$: $s = 2\sin\pi + \sin 2\pi = 0$

$t = -\pi$: $s = 2\sin(-\pi) + \sin(2(-\pi)) = 0$

$t = \frac{\pi}{3}$: $s = 2\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right) = \sqrt{3} + \frac{\sqrt{3}}{2} = \ldots$

$t = -\frac{\pi}{3}$: $s = 2\sin\left(-\frac{\pi}{3}\right) + \sin\left(2\left(-\frac{\pi}{3}\right)\right) = -\sqrt{3} - \frac{\sqrt{3}}{2} = \ldots$

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#22. curve $y = 2\sin\left(\pi \frac{x}{2}\right)$ $\left(\frac{1}{3}, 1\right)$

$x' = \sqrt{10}$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$\begin{matrix} x_1 & y_1 \end{matrix}$
$(0, 0)$

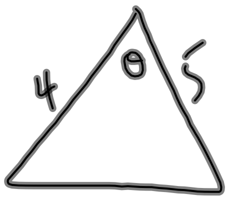
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d^2 = (x)^2 + (y)^2$$

$y = 2\sin\left(\pi \frac{x}{2}\right)$

$$d^2 = x^2 + \left[2\sin\left(\pi \frac{x}{2}\right)\right]^2$$

29.



$\frac{d\theta}{dt} = 0.06$ rad/s

$\theta = \frac{\pi}{3}$

$$A = \frac{1}{2}(\text{side})(\text{side})\sin(\text{contained } \angle)$$

$$A = \frac{1}{2}(4)(5)\sin\theta$$

$$A = 10\sin(\theta)$$