3.3 Derivatives and the Shapes of Graphs

hypercritical number:

a # for which $f''(x) = 0$
or $f''(x)$ is undefined

<u>Concavity</u>: refers to curvature of a graph.

Concave up $\cup$, $f''(x) > 0$

Concave down $\cap$, $f''(x) < 0$

<u>Inflection point</u>: If $f''(c) = 0$ and the graph changes concavity at c, then $(c, f(c))$ is a point of inflection

Ex: Do a complete analysis of

$$y = x^3 + 3x^2 - 9x - 11$$

(a) state domain and y-intercept

D: $\{x \mid x \in \mathbb{R}\}$ y-int $= (0,-11)$

R: $\{y \mid y \in \mathbb{R}\}$

(b) Algebraically determine the x-intercepts

solve: $x^3 + 3x^2 - 9x - 11 = 0$

P.I.R. $\pm\{1, 11\}$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -9 & -11 \\ & & -1 & -2 & 11 \\ \hline & 1 & 2 & -11 & \boxed{0} \end{array}$$

so $x = -1$ is a root (x-int)

now solve $x^2 + 2x - 11 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-11)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 + 44}}{2}$$

$$x = \frac{-2 \pm 4\sqrt{3}}{2}$$

$$x = -1 \pm 2\sqrt{3}$$

$x = -1 + 2\sqrt{3}$ $\boxed{= 2.46}$

$x = -1 - 2\sqrt{3}$ $\boxed{= -4.46}$

3 x-intercepts are $-1, 2.46, -4.46$

(c) Determine y'

$$y = x^3 + 3x^2 - 9x - 11$$

$$y' = 3x^2 + 6x - 9$$

(d) critical #'s $\Rightarrow y' = 0$

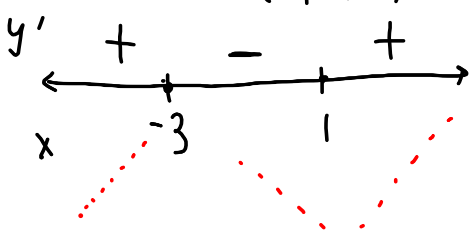
$$3x^2 + 6x - 9 = 0$$

$$x^3 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$$x = -3 \text{ OR } x = 1$$

(e) Intervals of increase / decrease.
(slope pos) (slope neg)



$y' = 3x^2 + 6x - 9$

Increasing on $(-\infty, -3)$ and $(1, \infty)$
Decreasing on $(-3, 1)$

(f) Since y' switches from positive to negative at $x = -3$, this is a local max.
$(-3, 16)$ $y = (-3)^3 + 3(-3)^2 - 9(-3) - 11$

Since y' switches from negative to positive at $x = 1$, this is a local min.
$(1, -16)$

(g) Determine y''

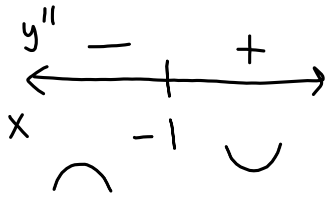
$$y' = 3x^2 + 6x - 9$$

$$y'' = 6x + 6$$

(h) Hypercritical #'s $\Rightarrow$ $y'' = 0$

$$6x + 6 = 0$$
$$x + 1 = 0$$
$$x = -1$$

(i) Intervals of concavity

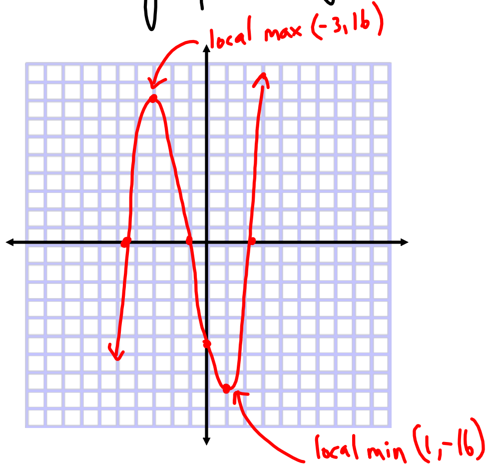


$y'' = 6x + 6$

Concave down on $(-\infty, -1)$
concave up on $(-1, \infty)$

Since concavity switches at $x = -1$ the point $(-1, 0)$ is a <u>point of inflection</u>.

(j) Draw the graph of $y = x^3 + 3x^2 - 9x - 11$



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#'s 21-26

Ex: Do a complete analysis of

$$f(x) = \frac{x^2}{x^2+1}$$

$x^2+1=0$
$x^2=-1$
No Real solutions

(a) Domain: $\{x \mid x \in \mathbb{R}\}$

y-int: $(0,0)$

(b) x-intercepts

$$0 = \frac{x^2}{(x^2+1)}$$

$$0 = x^2$$

$x = 0$ (Double)

(c) Calculate $f'(x)$

$$f(x) = \frac{x^2}{x^2+1}$$

$$f'(x) = \frac{(x^2+1)2x - x^2(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{\cancel{2x^3} + 2x \cancel{- 2x^3}}{(x^2+1)^2}$$

$$f'(x) = \frac{2x}{(x^2+1)^2}$$

(d) Critical #'s $\rightarrow$ $f'(x) = 0$

<u>OR</u> $f'(x) = \text{undefined}$

$$f'(x) = \frac{2x}{(x^2+1)^2}$$

$$0 = \frac{2x}{(x^2+1)^2}$$

$$0 = 2x$$

$$\boxed{0 = x}$$

<u>Note</u>: no NPV's

(e) intervals of increase/decrease.

x		0	
$f'(x)$	$-$		$+$

$$f'(1) = \frac{2(1)}{(1^2+1)^2} = +$$

decreasing on $(-\infty, 0)$

increasing on $(0, \infty)$

(f) local extrema:

since f' changes from neg. to pos. at $x = 0$, this is a local min.

(g) Determine $f''(x)$

$$f'(x) = \frac{2x}{(x^2+1)^2}$$

factor

$$f''(x) = \frac{(x^2+1)^2\, 2 - 2x\left(2(x^2+1)2x\right)}{\left((x^2+1)^2\right)^2}$$

$$f''(x) = \frac{\cancel{(x^2+1)}\left(2(x^2+1) - 8x^2\right)}{(x^2+1)^{\cancel{4}\,3}}$$

$$f''(x) = \frac{2-6x^2}{(x^2+1)^3} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

(h) Hypercritical #'s $\rightarrow$ $f''(x) = 0$ OR $f''(x) =$ undefined

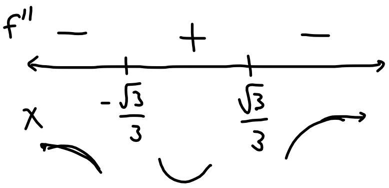
$$0 = \frac{2-6x^2}{(x^2+1)^3}$$

$$\Rightarrow \quad 0 = 2 - 6x^2$$

$$0 = 1 - 3x^2$$
$$3x^2 = 1$$
$$x^2 = \frac{1}{3}$$
$$x = \pm\frac{\sqrt{3}}{3}$$

Note $f''(x)$ is defined for all x

(i) Intervals of concavity

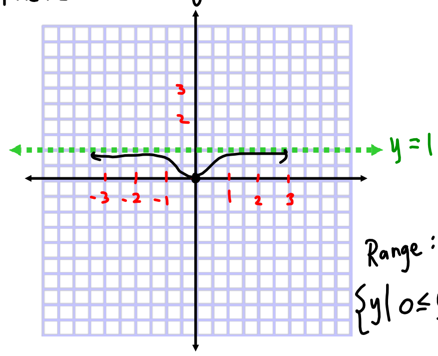


$$f(x) = \frac{x^2}{x^2+1}$$

$$f''(x) = \frac{2-6x^2}{(x^2+1)^3}$$

Note: $\lim\limits_{x\to\pm\infty} \frac{x^2}{x^2+1} = 1$, $\therefore$ H.A. at $y = 1$

(j) Sketch the graph:



Range: $\{y \mid 0 \le y < 1, y \in \mathbb{R}\}$

Practice Questions:

Page 151 #s 23-27, 29, 30, 35-40, 42, 51

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Symmetry

Even function: symmetry in y-axis

$$f(x) = f(-x)$$

Ex: $f(x) = \frac{x^2}{x^2+1}$ $\qquad f(-x) = \frac{(-x)^2}{(-x)^2+1}$

$$f(-x) = \frac{x^2}{x^2+1} = f(x)$$

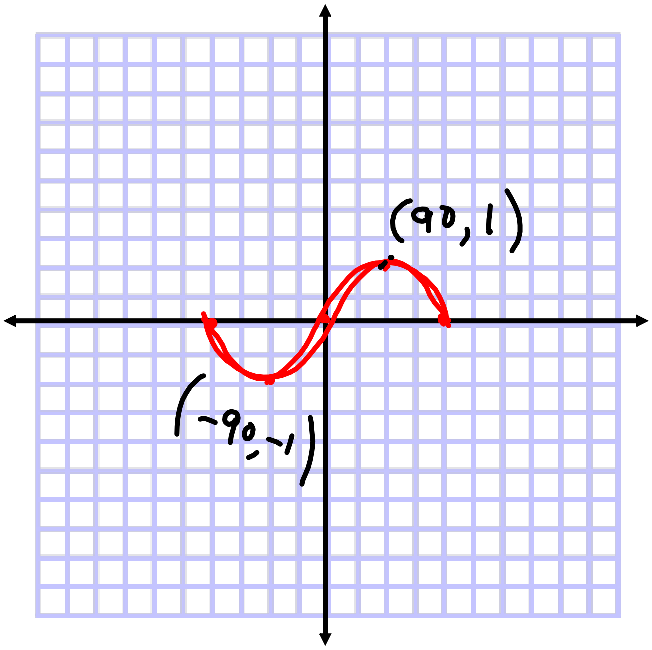
If (x, y) is on the graph, then $(-x, y)$ is also on the graph.

ODD FUNCTION: Symmetry in the origin

(Rotate 180° and get exact same graph)

$$f(x) = -f(-x)$$

If (x, y) is on the graph, then so too is $(-x, -y)$ on the graph


(90, 1)
(-90, -1)