UNIT 2: Limits and Continuity

Limit of a function as x approaches a particular value is the y-value that the function gets extremely close to (or sometimes becomes.)

Limit notation:

we say "the limit of f(x) as x approaches a"

we write $\boxed{\lim_{x \to a} f(x)}$

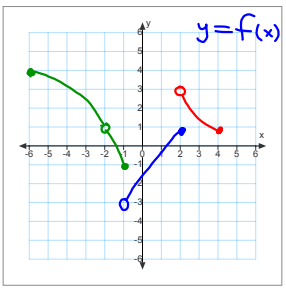
OR: limit as x approaches a of f(x)

Sometimes we are only interested in the limit as we approach x from one side or the other. These are called one-sided limits.

$\lim_{x \to a^-} f(x)$ = limit as $x \to a$ from the left (left hand limit)

$\lim_{x \to a^+} f(x)$ = limit as $x \to a$ from the right (Right hand limit)

Ex: Given graph of f(x)



① $\lim_{x \to -2^-} f(x) = 1$

② $\lim_{x \to -2^+} f(x) = 1$

③ $\lim_{x \to -2} f(x) = 1$

④ $\lim_{x \to -1^-} f(x) = -1$

⑤ $\lim_{x \to -1^+} f(x) = -3$

⑥ $\lim_{x \to -1} f(x) =$ DNE

⑦ $\lim_{x \to 2^-} f(x)$ 1

⑧ $\lim_{x \to 2^+} f(x) = 3$

⑨ $\lim_{x \to 2} f(x) =$ DNE

Note: If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

then $\lim_{x \to a} f(x) =$ DNE (Does Not Exist)

1.4 Evaluating Limits

Limit laws (p. 35)

Ex: $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

Ex: $\lim_{x \to a} (c \cdot f(x)) = c \cdot \lim_{x \to a} f(x)$, where c is some constant

Other limit rules

$\Rightarrow \lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ for $n \in N$

Ex: $\lim_{x \to 2} (3x)^2 = 36$ $\quad \left[\lim_{x \to 2} 3x\right]^2 = [6]^2 = 36$

$\Rightarrow \lim_{x \to a} c = c$ $\quad \Rightarrow \lim_{x \to a} x = a$

Some methods for evaluating limits:

Method 1: Direct Substitution

Ex: evaluate $\lim_{x \to 2} \frac{x^2 - 1}{x + 2}$

$$\Rightarrow \lim_{x \to 2} \frac{x^2 - 1}{x + 2} = \frac{2^2 - 1}{2 + 2} = \frac{3}{4}$$

Method 2: Approximate Substitution

Ex: Evaluate $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

sub in $x = 1.9, 1.99, 1.999$

$x = 2.1, 2.01, 2.001$

$\lim_{x \to 2^-}$ for $x = 1.9$ $\quad f(x) = \frac{1.9 - 2}{1.9^2 - 4} = 0.2564$

$f(1.99) = 0.2506$

$f(1.999) = 0.25006$

$\lim_{x \to 2^+}$ $\quad f(2.1) = 0.2439$

$f(2.01) = 0.2494$

$$\therefore \lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \frac{1}{4}$$

Method 3: Factor and Reduce

Note: Direct substitution does not work!

Ex: $\lim_{x \to 2} \frac{x-2}{x^2-4}$ $\qquad \frac{2-2}{2^2-4} = \frac{0}{0}$ indeterminate form

$$\Rightarrow \lim_{x \to 2} \frac{\cancel{x-2}}{\cancel{(x-2)}(x+2)}$$

$$= \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}$$

Method 4: Rationalize the Numerator

check direct subs.

Ex: Evaluate: $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$ $\qquad \boxed{\frac{0}{0}}$

$$\lim_{x \to 0} \frac{(\sqrt{x+1} - 1)}{(x)} \cdot \frac{(\sqrt{x+1} + 1)}{(\sqrt{x+1} + 1)}$$

multiply by the conjugate

$$\lim_{x \to 0} \frac{(x+1) - 1}{x(\sqrt{x+1} + 1)} = \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1} + 1)}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{\sqrt{0+1} + 1} = \frac{1}{2}$$

Examples: Evaluate the limit:

① $\lim_{x\to -3} \frac{-2x^2+1}{x^2+9} = \frac{-2(-3)^2+1}{(-3)^2+9} = \frac{-17}{18}$

② $\lim_{x\to 0^+} \frac{1}{x} = \infty$

③ $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

④ $\lim_{x\to 0} \frac{1}{x} = \text{DNE}$

Note: $x=0$ is a V.A.

⑤ $\lim_{x\to 6^-} \frac{|3x-18|}{6-x} = \lim_{x\to 6^-} \frac{3|x-6|}{6-x}$

Note: $|x-6| = -(x-6)$ for $x<6$

and since $x\to 6^-$ means $x<6$

our limit becomes: $\lim_{x\to 6^-} \frac{-3(x-6)}{6-x}$

$= \lim_{x\to 6^-} \frac{3\cancel{(6-x)}}{\cancel{6-x}} = 3$

⑥ $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$

⑦ $\lim_{x\to -4} \frac{\left(\frac{1}{4}+\frac{1}{x}\right)}{(4+x)} \cdot \frac{4x}{4x}$

$\lim_{x\to -4} \frac{\cancel{x+4}}{\cancel{(4+x)}4x}$

$\lim_{x\to -4} \frac{1}{4x} = -\frac{1}{16}$

24. $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$

$= \lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t(t+1)}\right)$

$= \lim_{t\to 0}\left(\frac{t+1}{t(t+1)}-\frac{1}{t(t+1)}\right)$

$= \lim_{t\to 0}\left(\frac{t}{t(t+1)}\right) = \lim_{t\to 0}\frac{1}{t+1} = \frac{1}{0+1} = 1$

$= \lim_{x\to 0}\left(\frac{1}{t+1}\right) = 1$ Page 43 # 1, 2, 5, 11-28, 44

28. $\lim_{h\to 0}\left(\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\right)$

$\lim_{h\to 0}\left(\frac{x^2-(x+h)^2}{hx^2(x+h)^2}\right)$

$\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}$

$\lim_{h\to 0}\frac{\cancel{x^2-x^2}-2xh-h^2}{hx^2(x+h)^2}$

$\lim_{h\to 0}\frac{-\cancel{h}(2x+h)}{\cancel{h}x^2(x+h)^2}$

$\lim_{h\to 0}\frac{-(2x+\cancel{h})}{x^2(x+\cancel{h})^2}$ $\cancel{h}\to 0$

$\frac{-2x}{x^4}$

$= \boxed{\frac{-2}{x^3}}$

more practice: p. 44 # 37, 38, 43, 44

1.5 Continuity

Definition: A function, f, is continuous at a number "a" if

$$\lim_{x \to a} f(x) = f(a)$$

To check for continuity at $x = a$

① $f(a)$ must exist

② $\lim_{x \to a} f(x)$ must exist

③ $\lim_{x \to a} f(x) = f(a)$

Ex: (i) Is $f(x) = \frac{x^2-1}{x-1}$ continuous at $x=1$?

condition ① $f(1) = \frac{(1)^2-1}{1-1} = \frac{0}{0}$ Does not Exist

$\therefore$ $f(x)$ is Not continuous at $x=1$

Ex: (ii) Is $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$ continuous at $x=3$?

condition 1 $f(3) = 5$ Exists ✓

condition 2 (Check Limit) $\lim_{x \to 3} f(x) \Rightarrow \lim_{x \to 3} \frac{x^2-9}{x-3}$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$$

$= 6$ Exists ✓

Condition ③ Is $\lim_{x \to 3} f(x) = f(3)$?

$6 \neq 5$ ☹ ✗

$\therefore$ $f(x)$ is NOT continuous at $x=3$

Ex ③ Is $f(x) = \begin{cases} \frac{x}{2}+1, & x \le 2 \\ 3-x, & x > 2 \end{cases}$

continuous at $x = 2$?

condition ① $f(2) = \frac{2}{2}+1 = 2$ Exists ✓

piece defined for $x \le 2$

condition ② $\lim_{x \to 2^-} \left(\frac{x}{2}+1\right) = \frac{2}{2}+1 = 2$

piece defined for $x > 2$

$\lim_{x \to 2^+} (3-x) = 3-2 = 1$

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, $\lim_{x \to 2} f(x)$ DNE

$\therefore$ $f(x)$ is Not continuous at $x = 2$

Ex ④: Is $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$ Continuous at $x = 3$?

C① $f(3) = 6$

C② $\lim_{x \to 3} f(x) = 6$

C③ $\lim_{x \to 3} f(x) = f(3)$

$\therefore$ $f(x)$ Is continuous at $x = 3$

NOTE: By redefining the function at $x = 3$, we have removed the discontinuity that existed in Ex(ii). So we say the discontinuity that existed in Ex(ii) is a Removable discontinuity

Ex 5: For what value(s) of x is this function not continuous?

$$f(x)=\begin{cases}\sqrt{3x-5}, & x\le 3\\ \dfrac{12}{9-x}, & 3<x\le 5\\ x^3-3x-6, & x>5\end{cases}$$

$3x-5\ge 0$

$x\ge \frac{5}{3}$

Note: function is not defined for $x<\frac{5}{3}$

Need to test for continuity at $x=3$ and $x=5$

Test $x=3$

1) $f(3)=\sqrt{3(3)-5}=\sqrt{4}=2$

2) check $\lim_{x\to 3} f(x)$

$$\begin{cases}\lim_{x\to 3^-}\sqrt{3x-5}=2\\ \lim_{x\to 3^+}\dfrac{12}{9-x}=2\end{cases}$$

$\therefore \lim_{x\to 3} f(x)=2$

3). $\lim_{x\to 3} f(x)=f(3)=2$

$\therefore f(x)$ IS continuous at $x=3$

Now Test $x=5$ for continuity

1.) $f(5)=\dfrac{12}{9-5}=3$ ✓

2.) $\lim_{x\to 5} f(x)$

$$\begin{cases}\lim_{x\to 5^-}\dfrac{12}{9-x}=3\\ \lim_{x\to 5^+}(x^3-3x-6)=5^3-3(5)-6=104\end{cases}$$

Since $\lim_{x\to 5^-} f(x)\neq \lim_{x\to 5^+} f(x)$,

$\lim_{x\to 5} f(x)$ does not EXIST

$\therefore f(x)$ is NOT continuous at $x=5$

Continuity from the left or right

- If $\lim_{x \to a^-} f(x) = f(a)$, then we say that $f(x)$ is continuous from the left at $x = a$

- If $\lim_{x \to a^+} f(x) = f(a)$, then we say that $f(x)$ is continuous from the right at $x = a$

Ex: Is $f(x) = \begin{cases} \frac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

continuous at $x = 2$?

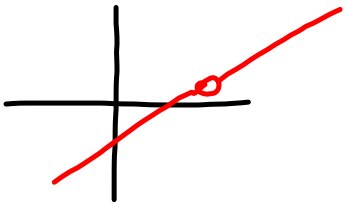
Check: $f(2) = \frac{2}{2} + 1 = 2$

$\lim_{x \to 2^-} f(x) = 2 \quad \lim_{x \to 2^+} f(x) = 1$

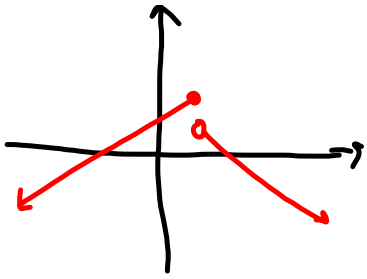
Since $\lim_{x \to 2} f(x)$ does not exist $f(x)$ is NOT continuous at $x = 2$. However, since $\lim_{x \to 2^-} f(x) = f(2)$ we can say that $f(x)$ is continuous from the left at $x = 2$. Also, since $\lim_{x \to 2^+} f(x) \neq f(2)$ we say $f(x)$ is Not continuous from the right at $x = 2$.

Types of discontinuities

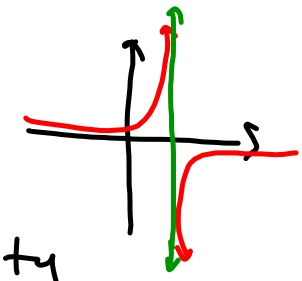
- point (removable) $\lim_{x \to a} f(x) =$ exists



- jump (non-removable) $\lim_{x \to a} f(x) =$ DNE



- infinite discontinuity (non-removable) $\lim_{x \to a} f(x) =$ DNE
  ↳ vertical asymptote at $x = a$



- Oscillating discontinuity