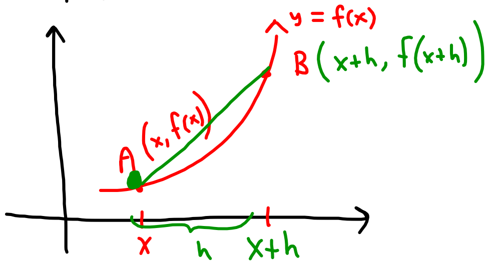
UNIT 4: Derivatives

Problem: How do we calculate the slope of the tangent line (i.e. using a single point)

one solution → approximate using two points (i.e slope of secant)

calculus solution → do the limit of the slopes of the secants as the two points move closer and closer together or as the distance between the two points approaches 0

Ex Determine the slope of the secant between A and B on the curve

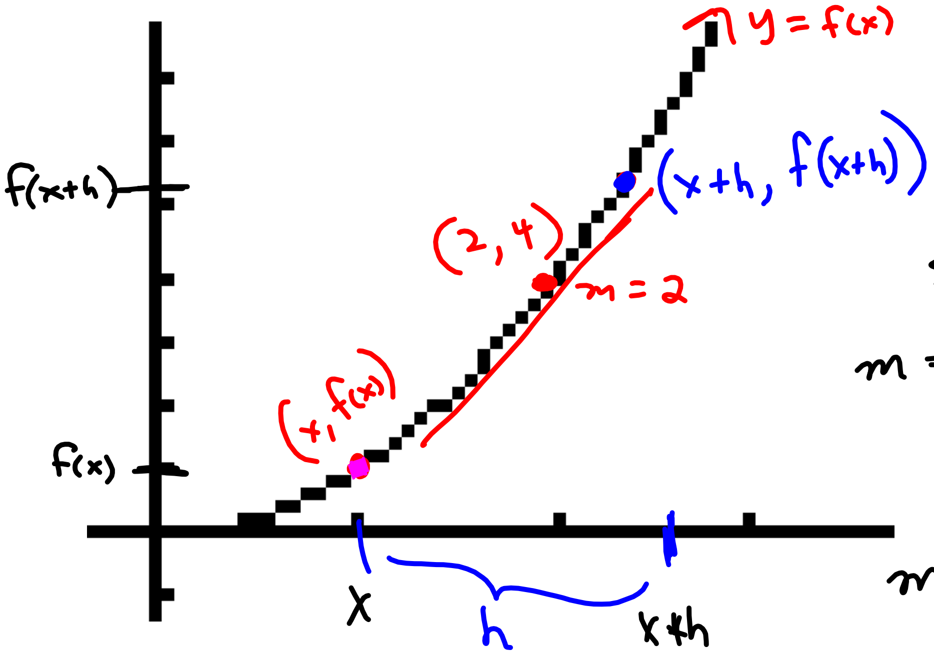


$$\underset{\text{of secant}}{\text{slope}} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x+h) - f(x)}{x+h - x}$$

$$\underset{\text{of secant}}{\text{Slope}} = \boxed{\frac{f(x+h) - f(x)}{h}}$$

Slope of the tangent line to a single point is the limit of this secant slope as $h \to 0$

i.e. $\boxed{\text{slope of tangent} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$



slope of secant

$$m = \frac{f(x+h) - f(x)}{x+h - x}$$

$$m = \frac{f(x+h) - f(x)}{h}$$

Slope of tangent is limit of slopes of secant lines

i.e. slope of tangent is

$$\boxed{m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Ex: Determine the slope of the tangent to $y = x^2$ at $x = 2$

$f(x) = x^2$

Solution:

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = x^2$

$f(x+h) = (x+h)^2$

$(x+h)(x+h) = x^2 + 2xh + h^2$

$$m = \lim_{h \to 0} \frac{(x+h)^2 - (x^2)}{h}$$

$$m = \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 \cancel{- x^2}}{h}$$

$$m = \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + \cancel{h}^{\,0})}{\cancel{h}}$$

$$m = 2x$$

So at $x = 2$, $m = 2(2) = 4$

Ex: Given the function $y = x^2$, determine the slope function for any point on the curve.

Solution:

$$\text{slope} = \lim_{h \to 0} \frac{[f(x+h)] - f(x)}{h}$$

Note: $f(x) = x^2$

$f(x+h) = (x+h)^2$

$= (x+h)(x+h)$

$= x^2 + xh + xh + h^2$

$$= \lim_{h \to 0} \frac{[(x+h)^2] - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + \cancel{h}^{\,0}$$

$$\underline{\text{Slope of tangent}} = 2x$$

$\hookrightarrow$ called **Derivative**

Definition of the derivative $(f'(x))$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Slope function for tangent to any point on a function/relation

Ex: Given $f(x) = 2x^2+1$, determine the derivative, $f'(x)$, using the definition.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\left[2(x+h)^2+1\right]-\left[2x^2+1\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[2(x^2+2xh+h^2)+1\right]-2x^2-1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2}+4xh+2h^2\cancel{+1}-\cancel{2x^2}\cancel{-1}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(4x+\cancel{2h}^{\to 0})}{\cancel{h}} = 4x$$

Ex ③ Given $f(x) = x^3 + 1$, determine the slope of the line tangent to the curve at $x = 2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^3 + 1] - [x^3 + 1]}{h}$$

$(x+h)(x+h)(x+h)$
$(x+h)(x^2+2xh+h^2)$
$x^3+2x^2h+xh^2$
$x^2h+2xh^2+h^3$
$x^3+3x^2h+3xh^2+h^3$

$$f'(x) = \lim_{h\to 0} \frac{[\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{1}] - \cancel{x^3} - \cancel{1}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh^{\to 0} + h^{2 \to 0})}{\cancel{h}}$$

$$f'(x) = 3x^2$$

slope of tangent to ANY point on curve

At $x = 2$: $f'(2) = 3(2)^2 = 12$

Ex ④: Determine the equation of the line tangent to the curve $f(x) = 2x^2 + x + 1$, at $x = -1$

$$m = f'(x) = \lim_{h\to 0} \frac{[f(x+h)] - [f(x)]}{h}$$

$$= \lim_{h\to 0} \frac{[2(x+h)^2 + (x+h) + 1] - [2x^2 + x + 1]}{h}$$

$$= \lim_{h\to 0} \frac{[2(x^2+2xh+h^2) + x + h + 1] - 2x^2 - x - 1}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h + \cancel{1} - \cancel{2x^2} - \cancel{x} - \cancel{1}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(4x + 2h^{\to 0} + 1)}{\cancel{h}}$$

$m = f'(x) = 4x + 1$

need $f'(-1) = 4(-1) + 1 = -3$ = slope of tangent $m = -3$

also need $f(-1) = 2(-1)^2 + (-1) + 1 = 2$, $(-1, 2)$

So, $(-1, 2)$ is point on the tangent line.

tangent line $y = -3x + b$

$2 = -3(-1) + b$

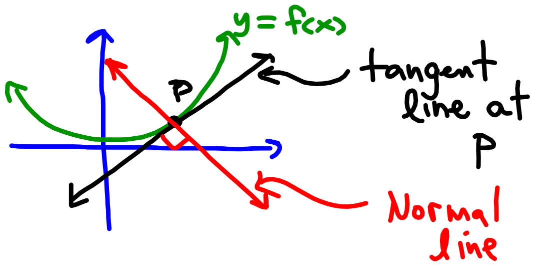
$-1 = b$

$\boxed{y = -3x - 1}$

$y - 2 = -3(x + 1)$

The Normal Line:

Line through curve at point of tangency which is $\perp$ to the tangent line.



Ex: Determine the equation of the tangent and the normal line to the graph of

$$y = \frac{1}{x+2} \quad \text{at } x = -1 \qquad (-1, 1)$$

$$m = f'(x) = \lim_{h\to 0} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{h}$$

$$= \lim_{h\to 0} \frac{\boxed{\frac{1}{x+h+2}} - \boxed{\frac{1}{x+2}}}{h} \qquad \frac{((x+2)(x+h+2))}{(x+2)(x+h+2)}$$

$$= \lim_{h\to 0} \frac{(x+2)-(x+h+2)}{h(x+2)(x+h+2)}$$

$$= \lim_{h\to 0} \frac{\cancel{x}\cancel{+2}-\cancel{x}-h-\cancel{2}}{h(x+2)(x+h+2)}$$

$$m = f'(x) = \lim_{h\to 0} \frac{-1}{(x+2)(x+\underset{0}{\cancel{h}}+2)} = \frac{-1}{(x+2)^2}$$

So, slope of tangent at $x=-1$ is $\frac{-1}{(-1+2)^2} = -1$

slope of normal line is ($\perp$ to tangent) so neg recip. of (-1) which is 1.

pt. of tangency is $(-1, 1)$

tangent: $y-1 = -1(x+1)$

Normal: $y-1 = 1(x+1)$

- [redacted] determine the derivative for each of the following functions using $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$:

(i) $f(x) = x^2 + 2$

(ii) $f(x) = x^3 - x^2$ $f'(x)$

(iii) $P(x) = \frac{1}{x+2}$ $p'(k)$

(iv) $S(t) = \frac{1-2t}{3t+4}$ $s'(t)$

(v) $f(x) = \sqrt{4+3x}$ $f'(x)$

(iv) $S(t) = \dfrac{1-2t}{3t+4}$

$$S'(t) = \lim_{h\to 0} \frac{S(t+h) - S(t)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1-2(t+h)}{3(t+h)+4} - \dfrac{1-2t}{3t+4}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{1-2t-2h}{3t+3h+4} - \dfrac{1-2t}{3t+4}\right)}{(h)} \left(\frac{(3t+3h+4)(3t+4)}{(3t+3h+4)(3t+4)}\right)$$

$$= \lim_{h\to 0} \frac{(1-2t-2h)(3t+4) - (1-2t)(3t+3h+4)}{h(3t+3h+4)(3t+4)}$$

$$= \lim_{h\to 0} \frac{(3t+4-6t^2-8t-6ht-8h) - (3t+3h+4-6t^2-6ht-8t)}{h(3t+3h+4)(3t+4)}$$

$$= \lim_{h\to 0} \frac{\cancel{3t}+\cancel{4}-\cancel{6t^2}-\cancel{8t}-\cancel{6ht}-8h-\cancel{3t}-3h-\cancel{4}+\cancel{6t^2}+\cancel{6ht}+\cancel{8t}}{h(3t+3h+4)(3t+4)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-11)}{\cancel{h}(3t+3h+4)(3t+4)} = \lim_{h\to 0} \frac{-11}{(3t+\underbrace{3h}_{0}+4)(3t+4)}$$

$$S'(t) = \frac{-11}{(3t+4)^2}$$

(v) $f(x)=\sqrt{4+3x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{4+3(x+h)} - \sqrt{4+3x}}{h}$$

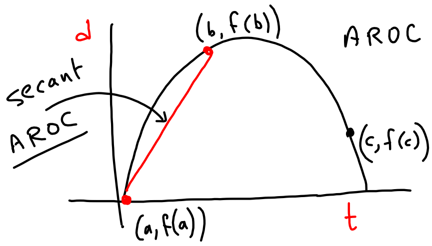
$$= \lim_{h\to 0} \left(\frac{\sqrt{4+3x+3h} - \sqrt{4+3x}}{h}\right)\frac{\left(\sqrt{4+3x+3h} + \sqrt{4+3x}\right)}{\left(\sqrt{4+3x+3h} + \sqrt{4+3x}\right)}$$

$$= \lim_{h\to 0} \frac{(4+3x+3h) - (4+3x)}{h\left(\sqrt{4+3x+3h} + \sqrt{4+3x}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{4} + \cancel{3x} + 3\cancel{h} - \cancel{4} - \cancel{3x}}{\cancel{h}\left(\sqrt{4+3x+\underbrace{3h}_{\to 0}} + \sqrt{4+3x}\right)}$$

$$f'(x) = \frac{3}{2\sqrt{4+3x}}$$

Rate of Change



$$AROC = \frac{f(b) - f(a)}{b - a}$$

IROC
= lim of slope of secants

Average rate of change = slope of secant
AROC

Instantaneous rate of change = slope of tangent
IROC

Ex: Given an object thrown into the air which follows a path defined by $h(t) = -4.9t^2 + 10t + 2$. What is the average rate of change between $t = 0$ and $t = 1$?

$$m_{avg} = \frac{h(1) - h(0)}{1 - 0}$$

$h(1) = -4.9(1)^2 + 10(1) + 2$
$h(1) = 7.1$

$h(0) = 2$

$$m_{avg} = \frac{7.1 - 2}{1} = 5.1 \text{ m/s}$$

Ex② Given same function as in Ex①, what is the instantaneous rate of change at $t = 1$ sec?

$$m_{ins} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \qquad \boxed{t = 1}$$

$$= \lim_{h \to 0} \frac{\left[-4.9(1+h)^2 + 10(1+h) + 2\right] - \left[-4.9(1)^2 + 10(1) + 2\right]}{h}$$

$$= \lim_{h \to 0} \frac{\left[-4.9(1 + 2h + h^2) + 10 + 10h + 2\right] - \left[-4.9 + 10 + 2\right]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-4.9} - 9.8h - 4.9h^2 \cancel{+10} + 10h \cancel{+2} \cancel{+4.9} \cancel{-10} \cancel{-2}}{h}$$

$$= \lim_{h \to 0} \frac{-9.8h - 4.9h^2 + 10h}{h} \qquad 0.2h$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-4.9\overset{0}{\cancel{h}} + 0.2)}{\cancel{h}}$$

$= 0.2$ m/s

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